mathtutor
algebra

Logarithms

Logarithms appear in all sorts of calculations in engineering and science, business and economics. Before the days of calculators they were used to assist in the process of multiplication by replacing the operation of multiplication by addition. Similarly, they enabled the operation of division to be replaced by subtraction. They remain important in other ways, one of which is that they provide the underlying theory of the logarithm function. This has applications in many fields, for example, the decibel scale in acoustics.

In order to master the techniques explained here it is vital that you do plenty of practice exercises so that they become second nature.

After reading this text and / or viewing the video tutorial on this topic you should be able to:

- explain what is meant by a logarithm
- state and use the laws of logarithms
- solve simple equations requiring the use of logarithms.

Contents

1. Introduction

In this unit we are going to be looking at logarithms. However, before we can deal with logarithms we need to revise indices. This is because logarithms and indices are closely related, and in order to understand logarithms a good knowledge of indices is required.

We know that

$$16 = 2^4$$

Here, the number 4 is the **power**. Sometimes we call it an **exponent**. Sometimes we call it an **index**. In the expression 2^4, the number 2 is called the **base**.

Example

We know that $64 = 8^2$.

In this example 2 is the power, or exponent, or index. The number 8 is the base.

2. Why do we study logarithms ?

In order to motivate our study of logarithms, consider the following:

we know that $16 = 2^4$. We also know that $8 = 2^3$

Suppose that we wanted to multiply 16 by 8.

One way is to carry out the multiplication directly using long-multiplication and obtain 128. But this could be long and tedious if the numbers were larger than 8 and 16. Can we do this calculation another way using the powers ? Note that

$$16 \times 8 \quad \text{can be written} \quad 2^4 \times 2^3$$

This equals

$$2^7$$

using the rules of indices which tell us to add the powers 4 and 3 to give the new power, 7. What was a multiplication sum has been reduced to an addition sum.

Similarly if we wanted to divide 16 by 8:

$$16 \div 8 \quad \text{can be written} \quad 2^4 \div 2^3$$

This equals

$$2^1 \quad \text{or simply} \quad 2$$

using the rules of indices which tell us to subtract the powers 4 and 3 to give the new power, 1.

If we had a look-up table containing powers of 2, it would be straightforward to look up 2^7 and obtain $2^7 = 128$ as the result of finding 16×8.

Notice that by using the powers, we have changed a multiplication problem into one involving addition (the addition of the powers, 4 and 3). Historically, this observation led John Napier (1550-1617) and Henry Briggs (1561-1630) to develop **logarithms** as a way of replacing multiplication with addition, and also division with subtraction.

3. What is a logarithm ?

Consider the expression $16 = 2^4$. Remember that 2 is the base, and 4 is the power. An alternative, yet equivalent, way of writing this expression is $\log_2 16 = 4$. This is stated as 'log to base 2 of 16 equals 4'. We see that the logarithm is the same as the power or index in the original expression. It is the base in the original expression which becomes the base of the logarithm.

The two statements

$$16 = 2^4 \qquad\qquad \log_2 16 = 4$$

are equivalent statements. If we write either of them, we are automatically implying the other.

Example

If we write down that $64 = 8^2$ then the equivalent statement using logarithms is $\log_8 64 = 2$.

Example

If we write down that $\log_3 27 = 3$ then the equivalent statement using powers is $3^3 = 27$.

So the two sets of statements, one involving powers and one involving logarithms are equivalent. In the general case we have:

Key Point

$$\text{if } x = a^n \qquad \text{then equivalently} \qquad \log_a x = n$$

Let us develop this a little more.

Because $10 = 10^1$ we can write the equivalent logarithmic form $\log_{10} 10 = 1$.
Similarly, the logarithmic form of the statement $2^1 = 2$ is $\log_2 2 = 1$.

In general, for any base a, $a = a^1$ and so $\log_a a = 1$.

Key Point

$$\log_a a = 1$$

We can see from the Examples above that indices and logarithms are very closely related. In the same way that we have rules or laws of indices, we have **laws of logarithms**. These are developed in the following sections.

4. Exercises

1. Write the following using logarithms instead of powers

a)	$8^2 = 64$	b)	$3^5 = 243$	c)	$2^{10} = 1024$	d)	$5^3 = 125$
e)	$10^6 = 1000000$	f)	$10^{-3} = 0.001$	g)	$3^{-2} = \frac{1}{9}$	h)	$6^0 = 1$
i)	$5^{-1} = \frac{1}{5}$	j)	$\sqrt{49} = 7$	k)	$27^{2/3} = 9$	l)	$32^{-2/5} = \frac{1}{4}$

2. Determine the value of the following logarithms

a)	$\log_3 9$	b)	$\log_2 32$	c)	$\log_5 125$	d)	$\log_{10} 10000$
e)	$\log_4 64$	f)	$\log_{25} 5$	g)	$\log_8 2$	h)	$\log_{81} 3$
i)	$\log_3 \left(\frac{1}{27}\right)$	j)	$\log_7 1$	k)	$\log_8 \left(\frac{1}{8}\right)$	l)	$\log_4 8$
m)	$\log_a a^5$	n)	$\log_c \sqrt{c}$	o)	$\log_s s$	p)	$\log_e \left(\frac{1}{e^3}\right)$

5. The first law of logarithms

Suppose

$$x = a^n \qquad \text{and} \qquad y = a^m$$

then the equivalent logarithmic forms are

$$\log_a x = n \qquad \text{and} \qquad \log_a y = m \tag{1}$$

Using the first rule of indices

$$xy = a^n \times a^m = a^{n+m}$$

Now the logarithmic form of the statement $xy = a^{n+m}$ is $\log_a xy = n + m$. But $n = \log_a x$ and $m = \log_a y$ from (1) and so putting these results together we have

$$\log_a xy = \log_a x + \log_a y$$

So, if we want to multiply two numbers together and find the logarithm of the result, we can do this by adding together the logarithms of the two numbers. This is the **first law**.

Key Point

$$\log_a xy = \log_a x + \log_a y$$

6. The second law of logarithms

Suppose $x = a^n$, or equivalently $\log_a x = n$. Suppose we raise both sides of $x = a^n$ to the power m:

$$x^m = (a^n)^m$$

Using the rules of indices we can write this as

$$x^m = a^{nm}$$

Thinking of the quantity x^m as a single term, the logarithmic form is

$$\log_a x^m = nm = m \log_a x$$

This is the **second law**. It states that when finding the logarithm of a power of a number, this can be evaluated by multiplying the logarithm of the number by that power.

Key Point

$$\log_a x^m = m \log_a x$$

7. The third law of logarithms

As before, suppose

$$x = a^n \qquad \text{and} \qquad y = a^m$$

with equivalent logarithmic forms

$$\log_a x = n \qquad \text{and} \qquad \log_a y = m \tag{2}$$

Consider $x \div y$.

$$\begin{aligned} \frac{x}{y} &= a^n \div a^m \\ &= a^{n-m} \end{aligned}$$

using the rules of indices.

In logarithmic form

$$\log_a \frac{x}{y} = n - m$$

which from (2) can be written

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

This is the **third law**.

Key Point

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

8. The logarithm of 1

Recall that any number raised to the power zero is 1: $a^0 = 1$. The logarithmic form of this is

$$\log_a 1 = 0$$

Key Point

$$\log_a 1 = 0$$

The logarithm of 1 in any base is 0.

9. Examples

Example

Suppose we wish to find $\log_2 512$.

This is the same as being asked 'what is 512 expressed as a power of 2 ?'
Now 512 is in fact 2^9 and so $\log_2 512 = 9$.

Example

Suppose we wish to find $\log_8 \frac{1}{64}$.

This is the same as being asked 'what is $\frac{1}{64}$ expressed as a power of 8 ?'
Now $\frac{1}{64}$ can be written 64^{-1}. Noting also that $8^2 = 64$ it follows that

$$\frac{1}{64} = 64^{-1} = (8^2)^{-1} = 8^{-2}$$

using the rules of indices. So $\log_8 \frac{1}{64} = -2$.

Example

Suppose we wish to find $\log_5 25$.

This is the same as being asked 'what is 25 expressed as a power of 5 ?'

Now $5^2 = 25$ and so $\log_5 25 = 2$.

Example

Suppose we wish to find $\log_{25} 5$.

This is the same as being asked 'what is 5 expressed as a power of 25 ?'

We know that 5 is a square root of 25, that is $5 = \sqrt{25}$. So $25^{\frac{1}{2}} = 5$ and so $\log_{25} 5 = \frac{1}{2}$.

Notice from the last two examples that by interchanging the base and the number

$$\log_{25} 5 = \frac{1}{\log_5 25}$$

This is true more generally:

Key Point

$$\log_b a = \frac{1}{\log_a b}$$

To illustrate this again, consider the following example.

Example

Consider $\log_2 8$. We are asking 'what is 8 expressed as a power of 2 ?' We know that $8 = 2^3$ and so $\log_2 8 = 3$.

What about $\log_8 2$? Now we are asking 'what is 2 expressed as a power of 8 ?' Now $2^3 = 8$ and so $2 = \sqrt[3]{8}$ or $8^{1/3}$. So $\log_8 2 = \dfrac{1}{3}$.

We see again

$$\log_8 2 = \frac{1}{\log_2 8}$$

10. Exercises

3 Each of the following expressions can be simplified to $\log N$.
Determine the value of N in each case. We have not explicitly written down the base. You can assume the base is 10, but the results are identical whichever base is used.

a)	$\log 3 + \log 5$	b)	$\log 16 - \log 2$	c)	$3 \log 4$
d)	$2 \log 3 - 3 \log 2$	e)	$\log 236 + \log 1$	f)	$\log 236 - \log 1$
g)	$5 \log 2 + 2 \log 5$	h)	$\log 128 - 7 \log 2$	i)	$\log 2 + \log 3 + \log 4$
j)	$\log 12 - 2 \log 2 + \log 3$	k)	$5 \log 2 + 4 \log 3 - 3 \log 4$	l)	$\log 10 + 2 \log 3 - \log 2$

11. Standard bases

There are two bases which are used much more commonly than any others and deserve special mention. These are

base 10 and base e

Logarithms to base 10, $\log_{10}$, are often written simply as log without explicitly writing a base down. So if you see an expression like $\log x$ you can assume the base is 10. Your calculator will be pre-programmed to evaluate logarithms to base 10. Look for the button marked log.

The second common base is e. The symbol e is called the **exponential constant** and has a value approximately equal to 2.718. This is a number like π in the sense that it has an infinite decimal expansion. Base e is used because this constant occurs frequently in the mathematical modelling of many physical, biological and economic applications. Logarithms to base e, $\log_e$, are often written simply as ln. If you see an expression like $\ln x$ you can assume the base is e. Such logarithms are also called **Naperian** or **natural** logarithms. Your calculator will be pre-programmed to evaluate logarithms to base e. Look for the button marked ln.

Key Point

Common bases:

$$\log \text{ means } \log_{10} \qquad \ln \text{ means } \log_e$$

where e is the exponential constant.

Useful results:

$$\log 10 = 1, \qquad \ln e = 1$$

12. Using logarithms to solve equations

We can use logarithms to solve equations where the unknown is in the power.

Suppose we wish to solve the equation $3^x = 5$. We can solve this by taking logarithms of both sides. Whilst logarithms to any base can be used, it is common practice to use base 10, as these are readily available on your calculator. So,

$$\log 3^x = \log 5$$

Now using the laws of logarithms, the left hand side can be re-written to give

$$x \log 3 = \log 5$$

This is more straightforward. The unknown is no longer in the power. Straightaway

$$x = \frac{\log 5}{\log 3}$$

If we wanted, this value can be found from a calculator.

Example

Solve $3^x = 5^{x-2}$. Again, notice that the unknown appears in the power. Take logs of both sides.

$$\log 3^x = \log 5^{x-2}$$

Now use the laws of logarithms.

$$x \log 3 = (x-2) \log 5$$

Notice now that the x we are trying to find is no longer in a power. Multiplying out the brackets

$$x \log 3 = x \log 5 - 2 \log 5$$

Rearrange this equation to get the two terms involving x on one side and the remaining term on the other side.

$$2 \log 5 = x \log 5 - x \log 3$$

Factorise the right-hand side by extracting the common factor of x.

$$\begin{aligned} 2 \log 5 &= x(\log 5 - \log 3) \\ &= x \log \left(\frac{5}{3}\right) \end{aligned}$$

using the laws of logarithms.

And finally

$$x = \frac{2 \log 5}{\log \left(\frac{5}{3}\right)}$$

If we wanted, this value can be found from a calculator.

13. Inverse operations

Suppose we pick a base, 2 say.

Suppose we pick a power, 8 say.

We will now raise the base 2 to the power 8, to give 2^8.

Suppose we now take logarithms to base 2 of 2^8.

We then have

$$\log_2 2^8$$

Using the laws of logarithms we can write this as

$$8 \log_2 2$$

Recall that $\log_a a = 1$, so $\log_2 2 = 1$, and so we have simply 8 again, the number we started with.

So, raising the base 2 to a power, and then finding the logarithm to base 2 of the result are inverse operations.

Let's look at this another way.

Suppose we pick a number, 8 say.

Suppose we find its logarithm to base 2, to evaluate $\log_2 8$.

Suppose we now raise the base 2 to this power: $2^{\log_2 8}$.

Because $8 = 2^3$ we can write this as $2^{\log_2 2^3}$. Using the laws of logarithms this equals $2^{3 \log_2 2}$ which equals 2^3 or 8, since $\log_2 2 = 1$. We see that raising the base 2 to the logarithm of a number to base 2 results in the original number.

So raising a base to a power, and finding the logarithm to that base are inverse operations. Doing one operation, and then following it by the other, we end up where we started.

Example
Suppose we are working in base e. We can pick a number x and evaluate e^x. If we follow this by taking logarithms to base e we obtain

$$\ln \mathrm{e}^x$$

Using the laws of logarithms this equals

$$x \ln \mathrm{e}$$

but $\ln \mathrm{e} = 1$ and so we are left with simply x again. So, raising the base e to a power, and then finding logarithms to base e are inverse operations.

Example
Suppose we are working in base 10. We can pick a number x and evaluate 10^x. If we follow this by taking logarithms to base 10 we obtain

$$\log 10^x$$

Using the laws of logarithms this equals

$$x \log 10$$

but $\log 10 = 1$ and so we are left with simply x again. So, raising the base 10 to a power, and then finding logarithms to base 10 are inverse operations.

Key Point

$$\ln \mathrm{e}^x = x, \qquad \mathrm{e}^{\ln x} = x$$

Similarly,

$$\log 10^x = x, \qquad 10^{\log x} = x$$

These results will be useful in doing calculus, especially in solving differential equations.

14. Exercises

4 Use logarithms to solve the following equations

a)	$10^x = 5$	b)	$\mathrm{e}^x = 8$	c)	$10^x = \frac{1}{2}$	d)	$\mathrm{e}^x = 0.1$
e)	$4^x = 12$	f)	$3^x = 2$	g)	$7^x = 1$	h)	$\left(\frac{1}{2}\right)^x = \frac{1}{100}$
i)	$\pi^x = 10$	j)	$\mathrm{e}^x = \pi$	k)	$\left(\frac{1}{3}\right)^x = 2$	l)	$10^x = \mathrm{e}^{2x-1}$

Answers to Exercises on Logarithms

1.

a)	$\log_8 64 = 2$	b)	$\log_3 243 = 5$	c)	$\log_2 1024 = 10$
d)	$\log_5 125 = 3$	e)	$\log_{10} 1000000 = 6$	f)	$\log_{10} 0.001 = -3$
g)	$\log_3 \left(\frac{1}{9}\right) = -2$	h)	$\log_6 1 = 0$	i)	$\log_5 \left(\frac{1}{5}\right) = -1$
j)	$\log_{49} 7 = \frac{1}{2}$	k)	$\log_{27} 9 = \frac{2}{3}$	l)	$\log_{32} \left(\frac{1}{4}\right) = -\frac{2}{5}$

2.

a)	2	b)	5	c)	3	d)	4
e)	3	f)	$\frac{1}{2}$	g)	$\frac{1}{3}$	h)	$\frac{1}{4}$
i)	-3	j)	0	k)	-1	l)	$\frac{3}{2}$
m)	5	n)	$\frac{1}{2}$	o)	1	p)	-3

3.

a)	15	b)	8	c)	64	d)	$\frac{9}{8}$
e)	236	f)	236	g)	800	h)	1
i)	24	j)	9	k)	$\frac{2592}{64} = \frac{81}{2}$	l)	45

4. All answers are correct to 3 decimal places

a)	0.699	b)	2.079	c)	-0.301	d)	-2.303
e)	1.792	f)	0.631	g)	0	h)	6.644
i)	2.011	j)	1.145	k)	-0.631	l)	-3.305